

# $W^{1,q}$ estimates for the extremal solution of reaction-diffusion problems

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## Abstract

We establish a new  $W^{1,2\frac{n-1}{n-2}}$  estimate for the extremal solution of  $-\Delta u = \lambda f(u)$  in a smooth bounded domain  $\Omega$  of  $\mathbb{R}^n$ , which is convex, for arbitrary positive and increasing nonlinearities  $f \in C^1(\mathbb{R})$  satisfying  $\lim_{t \rightarrow +\infty} f(t)/t = +\infty$ .

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## 1. Introduction

Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^n$  and consider the reaction-diffusion problem

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)_\lambda$$

where  $\lambda$  is a positive parameter and  $f$  is a positive and increasing  $C^1$  function satisfying

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty. \quad (1.2)$$

Crandall and Rabinowitz [7] proved, using the Implicit Function Theorem, the existence of an extremal parameter  $\lambda^* \in (0, +\infty)$  such that problem  $(1.1)_\lambda$  admits a classical minimal solution  $u_\lambda$  for all  $\lambda \in (0, \lambda^*)$ . Here, minimal means that it is smaller than any other nonnegative solution. Moreover,

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the least eigenvalue of the linearized operator at  $u_\lambda$ ,  $-\Delta - \lambda f'(u_\lambda)$ , is positive for all  $\lambda \in (0, \lambda^*)$ . Alternatively, this can be reached by using an iteration argument to obtain that  $u_\lambda$  is an absolute minimizer of the associated energy functional

$$J(u_\lambda) := \int_{\Omega} |\nabla u_\lambda|^2 - \lambda F(u_\lambda) dx, \quad (1.3)$$

in the convex set  $\{w \in H_0^1(\Omega) : 0 \leq w \leq u_\lambda\}$ , where  $F' = f$ . In particular,  $u_\lambda$  will be semi-stable in the sense that the second variation of energy at  $u_\lambda$  is nonnegative definite:

$$Q_{u_\lambda}(\varphi) := \int_{\Omega} |\nabla \varphi|^2 - \lambda f'(u_\lambda) \varphi^2 dx \geq 0 \quad \text{for all } \varphi \in C_0^1(\Omega). \quad (1.4)$$

Brezis *et al.* [1] proved that there is no weak solution for  $\lambda > \lambda^*$ , while the increasing limit

$$u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

is a weak solution of the extremal problem  $(1.1)_{\lambda^*}$ , *i.e.*,  $u^* \in L^1(\Omega)$ ,  $f(u^*) \text{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$ , and

$$\int_{\Omega} u^*(-\Delta \varphi) dx = \lambda \int_{\Omega} f(u^*) \varphi dx \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}).$$

This solution is known as *extremal solution* of the extremal problem  $(1.1)_{\lambda^*}$ .

The study of the regularity of the extremal solution started to grow after Brezis and Vázquez raised some open problems in [2]. In this direction, Nedev [10] proved, in an unpublished preprint, that  $u^* \in H_0^1(\Omega)$  for every positive and increasing nonlinearity  $f$  satisfying (1.2) when the domain is convex (see also Theorem 2.9 in [5]). The proof uses the Pohožaev identity and the fact that  $u_\lambda$  is an absolute minimizer of the functional  $J$ , defined in (1.3), on the compact set  $\{w \in H_0^1(\Omega) : 0 \leq w \leq u_\lambda\}$ , and hence,  $J(u_\lambda) \leq J(0) = 0$ .

Our main result establishes that  $u^* \in W_0^{1,2\frac{n-1}{n-2}}(\Omega)$  for any convex domain  $\Omega$  and any nonlinearity  $f$  satisfying the above assumptions. In particular,  $u^* \in H_0^1(\Omega)$ . We prove it using a geometric Sobolev inequality on the graph of minimal solutions  $u_\lambda$ .

**Theorem 1.1.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^n$  with  $n \geq 3$  and  $f$  a positive and increasing  $C^1$  function satisfying (1.2). Let  $u_\lambda \in C_0^2(\overline{\Omega})$  be the*

minimal solution of  $(1.1)_\lambda$  for  $\lambda \in (0, \lambda^*)$  and

$$I(t) := \int_{\{u_\lambda \geq t\}} (1 + |\nabla u_\lambda|^2)^{\frac{n-1}{n-2}} dx, \quad t \in (0, \|u_\lambda\|_{L^\infty(\Omega)}).$$

There exists a positive constant  $C$  depending only on  $n$  such that the following inequality holds

$$CI(t)^{2\frac{n-1}{n}} \leq \frac{1}{t^2} \left( \int_{\{u_\lambda \leq t\}} (1 + |\nabla u_\lambda|^2) |\nabla u_\lambda|^2 dx \right) I(t) + \left( \int_{\{u_\lambda = t\}} (1 + |\nabla u_\lambda|^2)^{\frac{1}{2}\frac{n-1}{n-2}} dS \right)^2 \quad (1.5)$$

for a.e.  $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$ .

If in addition  $\Omega$  is convex then the extremal solution  $u^* \in W_0^{1, 2\frac{n-1}{n-2}}(\Omega)$ .

In the last decade several authors studied the regularity of the extremal solution (see the monograph by Dupaigne [8] and references therein). However, there are few results for general reaction terms  $f$  (i.e., positive and increasing nonlinearities satisfying (1.2)). Cabré [4] established that  $u^* \in L^\infty(\Omega)$  when  $n \leq 4$  and the domain is convex. More recently, Cabré and the author [5] proved for  $n \geq 5$  that there exists a constant  $C$  depending only on  $n$  such that

$$\left( \int_{\{u_\lambda > t\}} (u_\lambda - t)^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{2n}} \leq \frac{C}{t} \left( \int_{\{u_\lambda \leq t\}} |\nabla u_\lambda|^4 dx \right)^{1/2}$$

for all  $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$ . As a consequence, it is proved that the extremal solution  $u^*$  belongs to  $L^{\frac{2n}{n-4}}(\Omega)$  when the domain is convex and the dimension  $n \geq 5$ . The first step in the proof of both results is to take  $\varphi = |\nabla u_\lambda| \eta$  as a test function in the semistability condition (1.4) and use the following geometric identity

$$(\Delta |\nabla u_\lambda| + \lambda f'(u_\lambda) |\nabla u_\lambda|) |\nabla u_\lambda| = \bar{A}^2 |\nabla u_\lambda|^2 + |\nabla_{\bar{T}} |\nabla u_\lambda||^2 \quad (1.6)$$

in  $\{x \in \Omega : |\nabla u_\lambda| > 0\}$ , where  $\bar{A}^2(x)$  denotes the second fundamental form at  $x$  of the  $(n-1)$ -dimensional hypersurface  $\{y \in \Omega : |u_\lambda(y)| = |u_\lambda(x)|\}$  and  $\nabla_{\bar{T}}$  is the tangential gradient with respect to this level set. Sternberg and Zumbrun [11, 12] made this choice to obtain

$$Q_{u_\lambda}(|\nabla u_\lambda| \eta) = \int_{\Omega \cap \{|\nabla u_\lambda| > 0\}} |\nabla u_\lambda|^2 |\nabla \eta|^2 - (\bar{A}^2 |\nabla u_\lambda|^2 + |\nabla_{\bar{T}} |\nabla u_\lambda||^2) \eta^2 dx$$

for every Lipschitz function  $\eta$  in  $\overline{\Omega}$  such that  $\eta|_{\partial\Omega} \equiv 0$ , where  $Q_{u_\lambda}$  is the quadratic form defined in (1.4). The second step in the proof is to choose an appropriate function  $\eta = \eta(u)$  and use the coarea formula and a Sobolev inequality on the  $(n-1)$ -dimensional hypersurface  $\{y \in \Omega : u_\lambda(y) = u_\lambda(x)\}$ .

The first ingredient in the proof of Theorem 1.1 is the following identity, analogue to (1.6), involving the second fundamental form of  $\text{Graph}(u_\lambda)$ .

**Proposition 1.2.** *Let  $u \in C_0^3(\overline{\Omega})$  be a positive function and  $v(x, x_{n+1}) := u(x) - x_{n+1}$  for all  $(x, x_{n+1}) \in \Omega \times \mathbb{R}$ . Let  $\nu = -\frac{\nabla v}{|\nabla v|} \in \mathbb{R}^{n+1}$  be the unit normal vector to  $\text{Graph}(u)$ ,  $A^2$  the second fundamental form of  $\text{Graph}(u)$ , and  $\nabla_T \varphi := \nabla \varphi - (\nu \cdot \nabla \varphi)\nu$  for every  $\varphi \in C^1(\mathbb{R}^{n+1})$ . The following identity holds*

$$(\Delta|\nabla v| + \nu \cdot \nabla \Delta v) |\nabla v| = A^2 |\nabla v|^2 + |\nabla_T |\nabla v||^2 \quad \text{in } \Omega. \quad (1.7)$$

In particular, if  $u \in C^2(\overline{\Omega})$  is a solution of (1.1) $_\lambda$  and  $f \in C^1(\mathbb{R})$  then

$$(\Delta|\nabla v| + \lambda f'(u) |\nabla v|) |\nabla v| = \lambda f'(u) + A^2 |\nabla v|^2 + |\nabla_T |\nabla v||^2 \quad \text{in } \Omega. \quad (1.8)$$

**Remark 1.3.** (i) Let  $u \in C^2(\overline{\Omega})$  be a solution of (1.1) $_\lambda$ . Note that

$$\Delta v = \sum_{i=1}^{n+1} v_{ii} = \sum_{i=1}^n u_{ii} = \Delta u$$

and

$$\nabla \Delta v = (\nabla \Delta u, 0) = (-\lambda f'(u) \nabla u, 0) \in \mathbb{R}^{n+1}.$$

(ii) Farina, Sciunzi, and Valdinoci [9] and Cesaroni, Novaga, and Valdinoci [6] recently used identity (1.6) to obtain one-dimensional symmetry of solutions to some reaction-diffusion equations. In this sense identity (1.8) could be useful by itself.

The main novelty in the proof of Theorem 1.1 is that we use a Sobolev inequality on the  $n$ -dimensional hypersurface

$$\text{Graph}(u_\lambda) = \{(x, x_{n+1}) \in \Omega \times \mathbb{R} : x_{n+1} = u_\lambda(x)\} \subset \mathbb{R}^{n+1},$$

instead on the level sets  $\{y \in \Omega : u_\lambda(y) = u_\lambda(x)\}$  of  $u_\lambda$  as in [4, 5], and the geometric identity (1.8). More precisely, define  $v_\lambda(x, x_{n+1}) := u_\lambda(x) - x_{n+1}$  for every  $\lambda \in (0, \lambda^*)$ . Taking  $\varphi = |\nabla v_\lambda| \eta$  in the semistability condition (1.4) and using identity (1.8), we obtain

$$\int_{\Omega} \left( \lambda f'(u_\lambda) + A^2 |\nabla v_\lambda|^2 + |\nabla_T |\nabla v_\lambda||^2 \right) \eta^2 dx \leq \int_{\Omega} |\nabla \eta|^2 |\nabla v_\lambda|^2 dx \quad (1.9)$$

for every Lipschitz function  $\eta$  in  $\overline{\Omega}$  such that  $\eta|_{\partial\Omega} \equiv 0$ . Choosing  $\eta = \min\{u_\lambda, t\}$  as a test function in (1.9) and using a geometric Sobolev inequality on the  $n$ -dimensional hypersurface  $\{(x, x_{n+1}) \in \text{Graph}(u_\lambda) : x_{n+1} \geq t\}$  (see Theorem 2.1 below) we prove inequality (1.5) in Theorem 1.1. The  $W^{1,2\frac{n-1}{n-2}}$ -estimate for the extremal solution follows from (1.5) and the convexity of the domain.

The paper is organized as follows. In section 2 we recall a Sobolev inequality on  $n$ -dimensional hypersurfaces with boundary and we prove the geometric identities established in Proposition 1.2. In section 3 we prove Theorem 1.1.

## 2. Geometric identities and inequalities. Proof of Proposition 1.2

The first ingredient in the proof of Theorem 1.1 is the following Sobolev inequality on  $n$ -dimensional hypersurfaces (see section 28.5.3 in [3]): *Let  $M \subset \mathbb{R}^{n+1}$  be a  $C^2$  immersed  $n$ -dimensional compact hypersurface with  $n \geq 2$ . There exists a constant  $C(n)$  depending only on the dimension  $n$  such that, for every  $\phi \in C^1(M)$  it holds*

$$C(n) \left( \int_M |\phi|^{\frac{n}{n-1}} dV \right)^{\frac{n-1}{n}} \leq \int_M (|H\phi| + |\nabla\phi|) dV + \int_{\partial M} |\phi| dS, \quad (2.1)$$

where  $H$  is the mean curvature of  $M$ .

Let  $p^* := np/(n-p)$  be the critical Sobolev exponent. Replacing  $\phi$  by  $\phi^\alpha$  in (2.1), with  $\alpha = 2^*/1^* = 2(n-1)/(n-2)$ , and using Hölder and Minkowski inequalities it is standard to obtain the following result.

**Theorem 2.1** ([3]). *Let  $M \subset \mathbb{R}^{n+1}$  be a  $C^2$  immersed  $n$ -dimensional compact hypersurface with  $n \geq 3$ . There exists a constant  $C = C(n)$  depending only on the dimension  $n$  such that, for every  $\phi \in C^1(M)$  it holds*

$$C \left( \int_M |\phi|^{2^*} dV \right)^{2\frac{n-1}{n}} \leq \left( \int_M |\phi|^{2^*} dV \right) \left( \int_M (|H\phi|^2 + |\nabla\phi|^2) dV \right) + \left( \int_{\partial M} |\phi|^{2\frac{n-1}{n-2}} dS \right)^2, \quad (2.2)$$

where  $H$  is the mean curvature of  $M$  and  $2^* = 2n/(n-2)$ .

The second ingredient is identity (1.8) in Proposition 1.2. Before to prove it let us introduce some notation. Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^n$ ,  $v \in C^2(\Omega \times \mathbb{R})$ , and

$$\nu(x, x_{n+1}) = -\frac{\nabla v}{|\nabla v|}(x, x_{n+1})$$

the unit normal vector to the level set of  $v$  passing throughout  $(x, x_{n+1}) \in \{|\nabla v| \neq 0\}$ . Recall that the eigenvalues of  $\nu$  are the  $n$  principal curvatures  $\kappa_1, \dots, \kappa_n$  of the level sets of  $v$  and zero. In particular, the second fundamental form  $A^2 := \kappa_1^2 + \dots + \kappa_n^2$  of the level sets of  $v$  is given by  $A^2 = \nu_j^i \nu_i^j$ , where as usual Einstein summation convention is used. We denote the gradient along the level sets of  $v$  by  $\nabla_T$ , *i.e.*,

$$\nabla_T \phi = \nabla \phi - (\nabla \phi \cdot \nu) \nu \quad \text{for any } \phi \in C^1(\mathbb{R}^{n+1}).$$

Let us prove the identities established in Proposition 1.2.

*Proof of Proposition 1.2.* Let  $u \in C_0^3(\overline{\Omega})$  be a positive function and define  $v(x, x_{n+1}) = u(x) - x_{n+1}$  for all  $x \in \Omega$ .

We claim that  $\nabla_T \log |\nabla v| = (D\nu)\nu$ . Indeed, noting that

$$\begin{aligned} -\frac{v_{ij}}{|\nabla v|} &= \frac{(\nu^i |\nabla v|)_j}{|\nabla v|} = \nu^i \nabla^j \log |\nabla v| + \nu_j^i \\ &= \nu^i \nabla_T^j \log |\nabla v| + (\nabla \log |\nabla v| \cdot \nu) \nu^j \nu^i + \nu_j^i \end{aligned}$$

and  $v_{ij} = v_{ji}$  for all  $i, j = 1, \dots, n+1$ , we obtain

$$\nu_j^i = \nu_i^j + \nu^j \nabla_T^i \log |\nabla v| - \nu^i \nabla_T^j \log |\nabla v| \quad \text{for all } i, j = 1, \dots, n+1.$$

We prove the claim multiplying the previous equality by  $\nu^j$  and noting that  $\nu_j^i \nu^j = 0$  for every  $j = 1, \dots, n+1$  and  $\nabla_T \log |\nabla v| \cdot \nu = 0$ .

Now, using  $\nu_j^i \nu_i^j = A^2$  and  $\nabla_T^j \log |\nabla v| = \nu_j^i \nu^i$ , we compute

$$\begin{aligned} \Delta |\nabla v| &= -(v_{ij} \nu^j)_i = -\nu \cdot \nabla \Delta v - v_{ij} \nu_i^j \\ &= -\nu \cdot \nabla \Delta v + (|\nabla v| \nu^i)_j \nu_i^j \\ &= -\nu \cdot \nabla \Delta v + |\nabla v| \nu_j^i \nu_i^j + |\nabla v|_j \nabla_T^j \log |\nabla v| \\ &= -\nu \cdot \nabla \Delta v + (A^2 + |\nabla_T \log |\nabla v||^2) |\nabla v| \end{aligned}$$

to obtain identity (1.7).

If  $u \in C^2(\overline{\Omega})$  is a solution of  $(1.1)_\lambda$  and  $f \in C^1(\mathbb{R})$ , then by standard regularity results for uniformly elliptic equations one has  $u \in C^3(\overline{\Omega})$ . From (1.7) and noting that

$$\nabla \Delta v = (-\lambda f'(u) \nabla u, 0) \quad \text{and} \quad \nu = \frac{1}{|\nabla v|}(-\nabla u, 1),$$

we obtain

$$\Delta |\nabla v| = -\lambda f'(u) \frac{|\nabla u|^2}{|\nabla v|} + (A^2 + |\nabla_T \log |\nabla v||^2) |\nabla v|$$

proving the proposition.  $\square$

### 3. Proof of Theorem 1.1

Let  $u_\lambda$  be the minimal solution of  $(1.1)_\lambda$  for  $\lambda \in (0, \lambda^*)$ . Choosing  $\varphi = \sqrt{1 + |\nabla u_\lambda|^2} \eta$  as a test function in the semistability condition (1.4) and using Proposition 1.2, we first obtain (1.9).

**Lemma 3.1.** *Assume that  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^n$  and  $f$  a positive and increasing  $C^1$  function satisfying (1.2). Let  $u_\lambda$  be the minimal solution of  $(1.1)_\lambda$  and  $v_\lambda(x, x_{n+1}) := u_\lambda(x) - x_{n+1}$  for  $\lambda \in (0, \lambda^*)$ . The following inequality holds*

$$\int_{\Omega} (\lambda f'(u_\lambda) + A^2 |\nabla v_\lambda|^2 + |\nabla_T |\nabla v_\lambda||^2) \eta^2 dx \leq \int_{\Omega} |\nabla v_\lambda|^2 |\nabla \eta|^2 dx \quad (3.1)$$

for every Lipschitz function  $\eta$  in  $\overline{\Omega}$  with  $\eta|_{\partial\Omega} \equiv 0$ , where  $A^2$  and  $\nabla_T$  are as in Proposition 1.2.

*Proof.* In order to improve the notation, let us denote  $u_\lambda = u$  and  $v_\lambda = v$  for  $\lambda \in (0, \lambda^*)$ . Choosing  $\varphi = |\nabla v| \eta$  as a test function in (1.4) and integrating by parts we get

$$\begin{aligned} 0 &\leq Q_u(|\nabla v| \eta) \\ &= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 + |\nabla v| \nabla |\nabla v| \cdot \nabla \eta^2 + |\nabla |\nabla v||^2 \eta^2 - \lambda f'(u) |\nabla v|^2 \eta^2 dx \\ &= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 - (\operatorname{div}(|\nabla v| \nabla |\nabla v|) - |\nabla |\nabla v||^2 + \lambda f'(u) |\nabla v|^2) \eta^2 dx \\ &= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 - (|\nabla v| \Delta |\nabla v| + \lambda f'(u) |\nabla v|^2) \eta^2 dx. \end{aligned}$$

Inequality (3.1) follows directly from identity (1.8).  $\square$

Finally, using Lemma 3.1 and the geometric Sobolev inequality established in Theorem 2.1 we prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $u_\lambda \in C_0^2(\overline{\Omega})$  be the minimal solution of (1.1) $_\lambda$  for  $\lambda \in (0, \lambda^*)$  and  $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$ . Define  $v_\lambda(x, x_{n+1}) = u_\lambda(x) - x_{n+1}$ . Let  $M_t := \{(x, x_{n+1}) \in \text{Graph}(u_\lambda) : x_{n+1} \geq t\}$  and  $dV = \sqrt{1 + |\nabla u_\lambda|^2} dx$  its element of volume.

We start by proving inequality (1.5). On the one hand, taking  $\eta = \min\{u_\lambda, t\}$  as a test function in (3.1), using that  $f$  is an increasing function, and  $H^2 = (\kappa_1 + \dots + \kappa_n)^2 \leq nA^2 = n(\kappa_1^2 + \dots + \kappa_n^2)$ , we obtain

$$\begin{aligned} \int_{M_t} \left( H^2 |\nabla v_\lambda| + |\nabla_T |\nabla v_\lambda|^{\frac{1}{2}}|^2 \right) dV &\leq \int_{\{u_\lambda \geq t\}} \left( nA^2 |\nabla v_\lambda|^2 + \frac{1}{4} |\nabla_T |\nabla v_\lambda|^{\frac{1}{2}}|^2 \right) dx \\ &\leq \frac{n}{t^2} \int_{\{u_\lambda \leq t\}} |\nabla v_\lambda|^2 |\nabla u_\lambda|^2 dx \end{aligned} \quad (3.2)$$

for all  $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$ .

Therefore, applying Theorem 2.1 with  $M = M_t$  and  $\phi = |\nabla v_\lambda|^{1/2}$ , we obtain

$$\begin{aligned} C \left( \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \right)^{2\frac{n-1}{n}} &\leq \frac{n}{t^2} \left( \int_{\{u_\lambda \leq t\}} |\nabla v_\lambda|^2 |\nabla u_\lambda|^2 dx \right) \left( \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \right) \\ &\quad + \left( \int_{\partial M_t} |\nabla v_\lambda|^{\frac{n-1}{n-2}} dS \right)^2, \end{aligned} \quad (3.3)$$

where  $C$  is a constant depending only on  $n$ . This is inequality (1.5).

Assume in addition that  $\Omega$  is convex. To prove that the extremal solution  $u^*$  belongs to  $W_0^{1, 2\frac{n-1}{n-2}}(\Omega)$  we only need to bound the integrals on  $\{u_\lambda \leq t\}$  and  $\partial M_t$ , for some  $t$ , by a constant independent of  $\lambda$  and then let  $\lambda$  tend to  $\lambda^*$ . The same argument was done in the proof of Theorem 2.7 [5]. However, for convinience to the reader we sketch the proof.

Since  $\Omega$  is convex, there exist positive constants  $\varepsilon$  and  $\gamma$  independent of  $\lambda$  such that

$$\|u_\lambda\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{1}{\gamma} \|u^*\|_{L^1(\Omega)} \quad \text{for all } \lambda < \lambda^*, \quad (3.4)$$

where  $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$  (see Proposition 4.3 [5] and references therein). Moreover, if  $\lambda^*/2 < \lambda < \lambda^*$ , then  $u_\lambda \geq u_{\lambda^*/2} > c \text{dist}(\cdot, \partial\Omega)$  for



some positive constant  $c$  independent of  $\lambda \in (\lambda^*/2, \lambda^*)$ . Therefore, letting  $t := c\varepsilon/2$ , we have  $\{x \in \Omega : u_\lambda(x) \leq t\} \subset \Omega_{\varepsilon/2} \subset \Omega_\varepsilon$ .

Note that  $u_\lambda$  is a solution of the linear equation  $-\Delta u_\lambda = h(x) := \lambda f(u_\lambda(x))$  in  $\Omega_\varepsilon$  and that, by (3.4),  $u_\lambda$  and the right hand side  $h$  are bounded in  $L^\infty(\Omega_\varepsilon)$  by a constant independent of  $\lambda$ . Hence, using interior and boundary estimates for the linear Poisson equation and (3.3), we deduce that

$$\left( \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \right)^{\frac{2}{n}} \leq C_1 \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV + C_2$$

for some constants  $C_1$  and  $C_2$  independent of  $\lambda$ .

Finally, noting that  $2(n-1)/n > 1$  (since  $n \geq 3$ ) and  $|\nabla u_\lambda| \leq |\nabla v_\lambda|$  we obtain

$$\int_{\{u_\lambda \geq t\}} |\nabla u_\lambda|^{\frac{n}{n-2}+1} dx \leq \int_{\{u_\lambda \geq t\}} |\nabla v_\lambda|^{\frac{n}{n-2}+1} dx = \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \leq C,$$

for some constant  $C$  independent of  $\lambda$ . Letting  $\lambda$  tend to  $\lambda^*$  in the previous inequality we conclude that  $u^* \in W_0^{1, 2\frac{n-1}{n-2}}(\Omega)$  proving the theorem.  $\square$

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